Quantum Simulation of Time-Dependent Hamiltonians and the convenient illusion of Hilbert Space

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today

Outline

1. Introduction
   - Time Independent Hamiltonian
   - Time Dependent Hamiltonian
2. Quantum Algorithm
   - Randomized Quantum Algorithm
3. Application: The Hollowness of Hilbert Space
   - Counting Physical States

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Applications

- Simulate the dynamics of many body quantum systems
- Subroutine in Quantum Algorithms

Simulating Quantum Systems

Problem Statement

Given a k-local bounded Hamiltonian $H = \sum_{i=1}^{L} H_i$, find a quantum circuit U_{sim} that, for any n-qubit initial state $|\psi(0)\rangle$, produces an approximation $|\psi_{\text{sim}}\rangle$ of the final state $|\psi(t)\rangle = U_H(t)\,|\psi(0)\rangle$ up to $\epsilon > 0$:

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$$\max_{|\psi(0)\rangle} \| U_H(t)\,|\psi(0)\rangle - U_{sim}(t)\,|\psi(0)\rangle \| = \| U_H(t) - U_{sim}(t) \| < \epsilon$$

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Solovay Kitaev Theorem

Let U be a unitary map that acts non-trivially on k= O(1) qubits and let M be an arbitrary finite universal set of one- and two-qubit quantum gates. U acting on k=O(1) qubits can be approximated to within ϵ by a universal set S using $\mathcal{O}(\log^c(1/\epsilon))$ one and two qubit quantum gates from S, for some constant c.

Each term $e^{-iH_j t/m}$ is a k-body unitary that can be efficiently approximated using the Solovay Kitaev theorem.

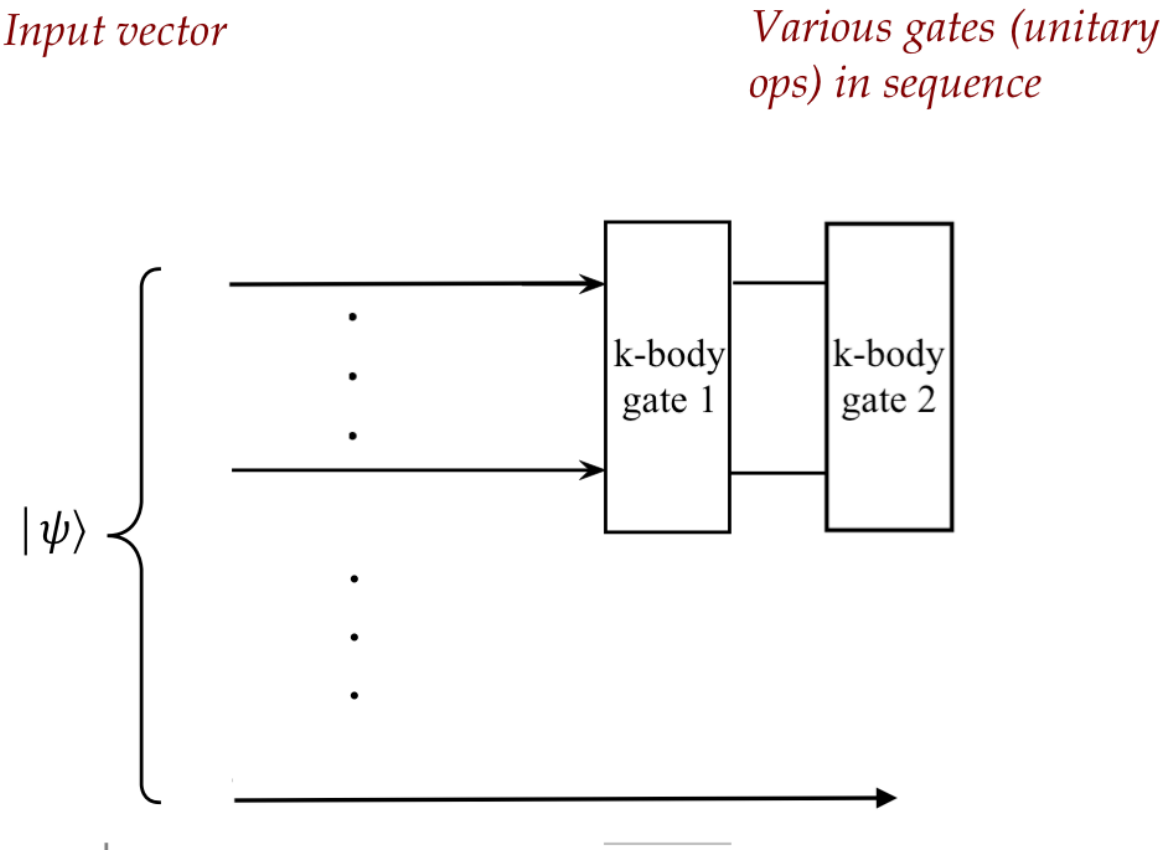
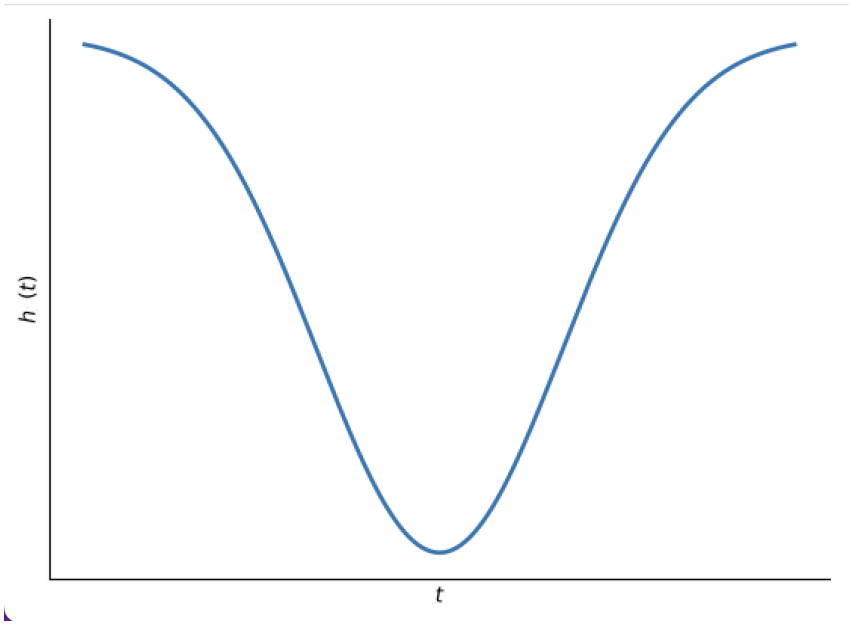


Figure: Each $e^{-iH_j t/m}$ acts on k registers

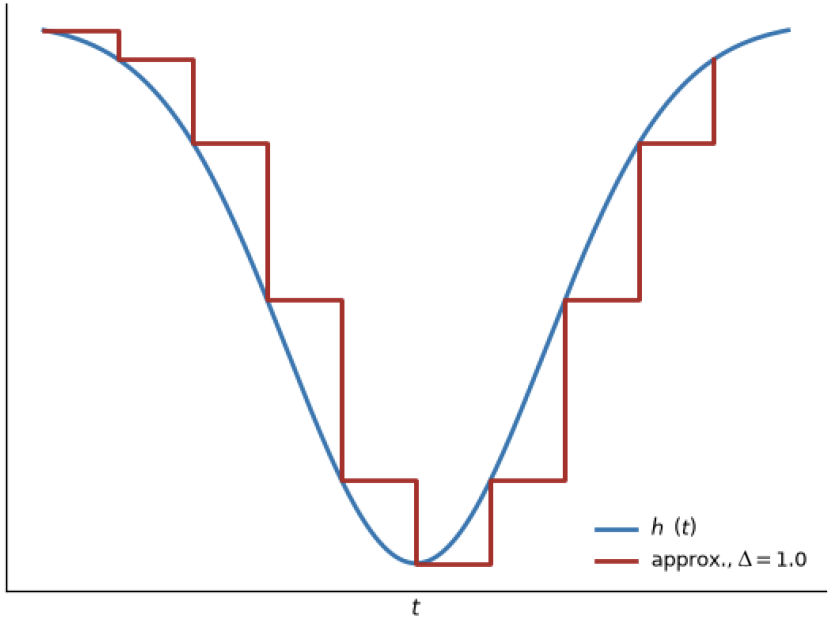
Time Dependent Hamiltonians

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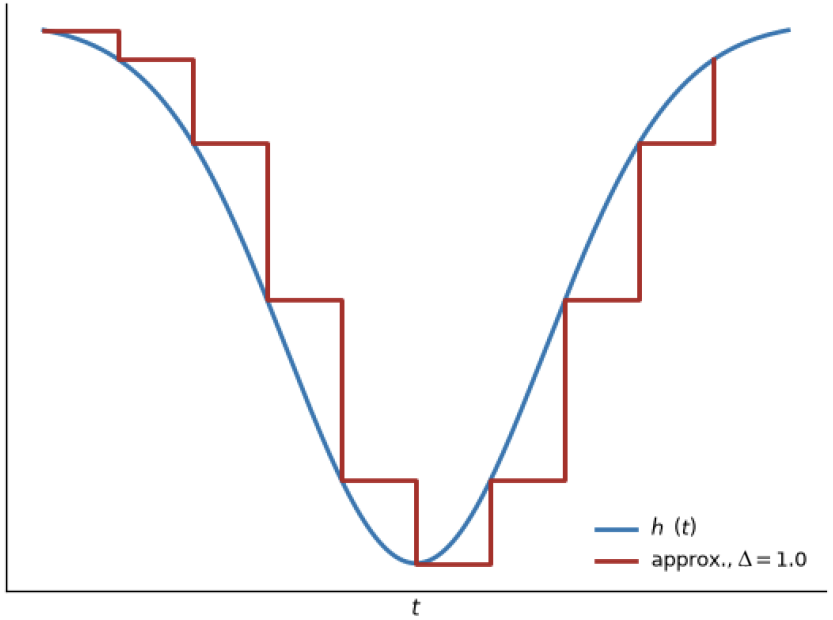
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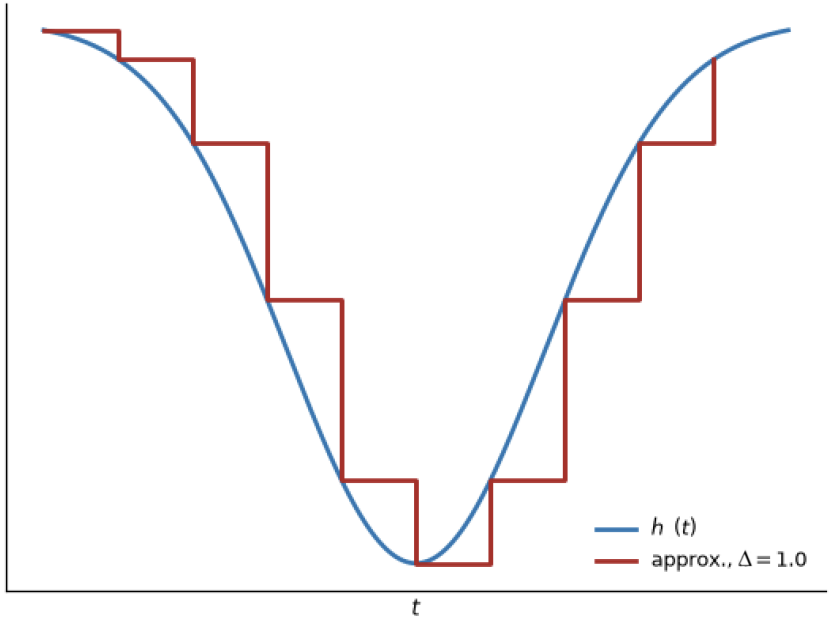


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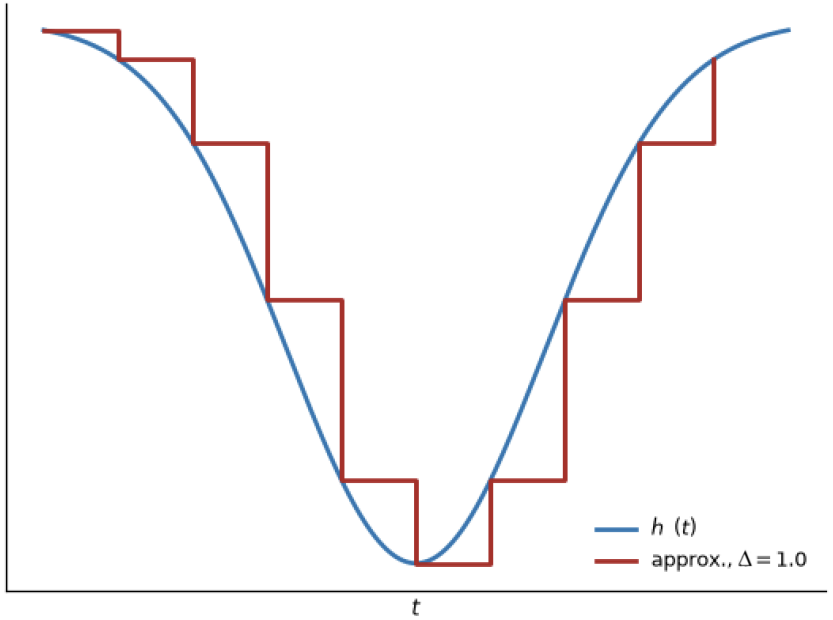
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- Rapidly changing Hamiltonian requires $\Delta t \sim \|\partial_t H\|^{-1}$

Challenge

Number of unitaries depends on $\|\partial_t H\|^{-1} \implies$ complexity grows

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Solution

Product Formula for Time Dependent Hamiltonians?

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Huyghebaert and Raedt(1990):

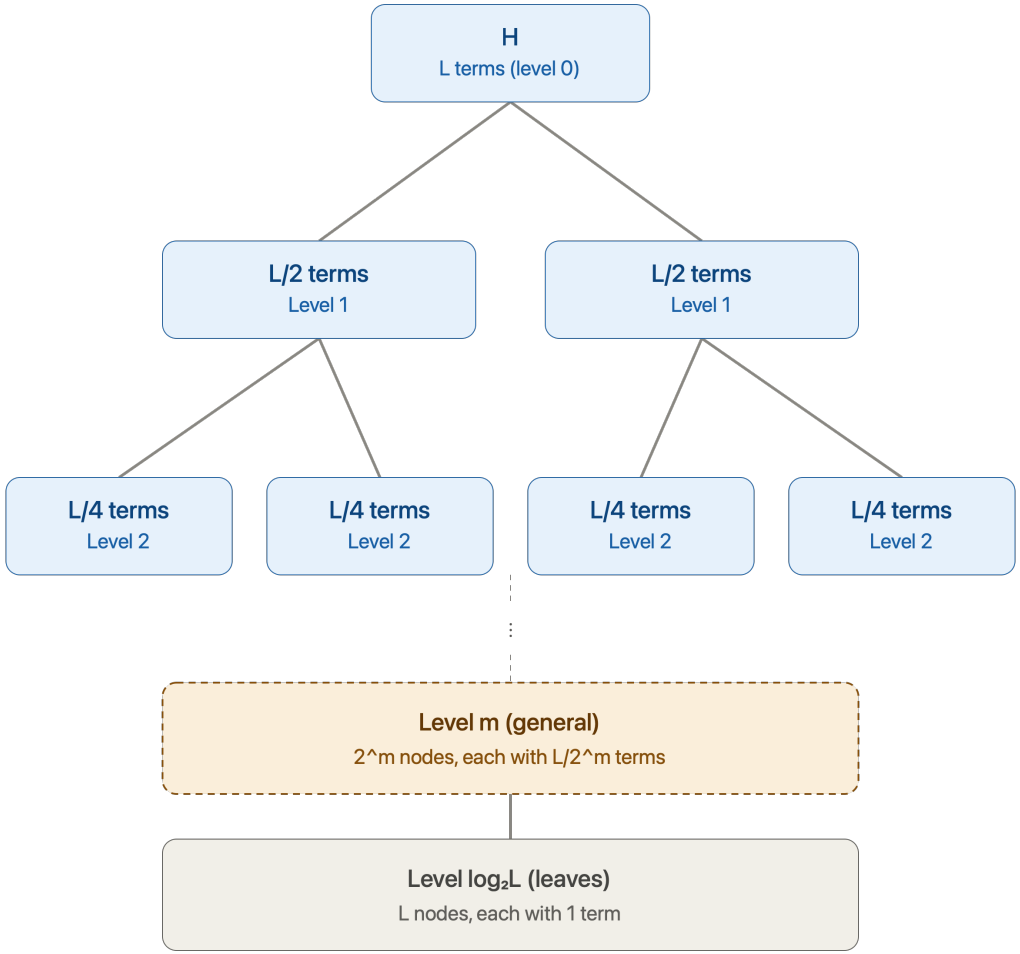
$$\left\| \mathcal{T} e^{-i\int_{t_j}^{t_j+\Delta t} H_1(t)+H_2(t)} - \mathcal{T} e^{-i\int_{t_j}^{t_j+\Delta t} H_1(t)} \mathcal{T} e^{-i\int_{t_j}^{t_j+\Delta t} H_2(t)} \right\| \leq c_{12}(\Delta t)^2$$

where $\boxed{c_{12} = \frac{1}{(\Delta t)^2}\int_{t_j}^{t_j+\Delta t} dv \int_{t_j}^{v} du \|[H_1(u), H_2(v)]\| \leq c_{\max}^2/2}$

with $c_{\max} = \max_t \max_X \|H_j(t)\|$

Recursion: from 2 termed to L termed Hamiltonian

- For $H = \sum_{j=1}^{L} H_j$ where L$\in$poly(n) where each term is a k-local, iterate the procedure $\log_2 L$ times.



Counting Gates

- $$\boxed{\text{Total error} \leq \frac{1}{2} c_{\max}^2 (\Delta t)^2 L^2}$$

This can be made arbitrarily small by choosing $\Delta t \sim (\#\text{qubits})^{\alpha}$.

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- Approximate time evolution for time T with a product of k-body unitaries such that total error is $\leq \epsilon/2$.

$$\Delta t = \frac{\epsilon}{T c_{\max}^2 L^2}$$

Counting Gates

- Total number of k-body Gates $G(\epsilon, T)$

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- Using Solovay Kitaev to approximate the complete time evolution with an error ϵ, we need total gates G_{tot}:

$$\boxed{G_{\text{tot}}(\epsilon, T) = d_{SK}\, G(\epsilon, T) \log^{c_{SK}}(G(\epsilon, T)/\epsilon)}$$

- $G_{\text{tot}}(\epsilon, T) \sim$ poly(# qubits, T)

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Time Interval Δt

$$\mathcal{T} e^{\left\{-i \int_0^T H_j(t')dt'\right\}} = \mathcal{T} e^{\left\{-i \int_{T-\Delta t}^T H_j(t')dt'\right\}} \times \ldots \times \mathcal{T} e^{\left\{-i \int_0^{\Delta t} H_j(t')dt'\right\}}$$

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Remove Time Order

$$\|\mathcal{T}e^{\left\{-i\int_0^{\Delta t} H_j(t')dt'\right\}} - e^{\left\{-i\int_0^{\Delta t} H_j(t')dt'\right\}}\| \leq 2\|H_j\|^2\Delta t^2$$

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Monte-Carlo Randomized Sampling

For $t_a \leftarrow (0, \Delta t)$: $\left\|\int_0^{\Delta t} H_j(t')dt' - \frac{1}{m}\sum_{a=1}^m H_j(t_a)\right\| \leq \frac{\Delta t\|H_j\|}{\sqrt{m}}$

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Trotter Decomposition

$$\|e^{-i\frac{\Delta t}{m}\sum_j H_j(t_j)} - \prod_j e^{-i\frac{\Delta t}{m}H_j(t_j)}\| \leq \frac{\Delta t^2\|H_j\|}{m}$$

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- Physical States seem to occupy a tiny volume.
  - They admit compact representations, unlike generic states in Hilbert Space.

Proposed Definition: Physical State

A state that can be reached in polynomial time, starting from a fiducial state (e.g. spin all up), under arbitrary local time-dependent Hamiltonian.

Counting Physical States

Corollary: A Counting Argument

$\mathrm{Vol}(\text{physical states}) \approx \#\text{poly-size quantum circuits} \times \mathrm{Vol}(\epsilon\text{-ball}).$

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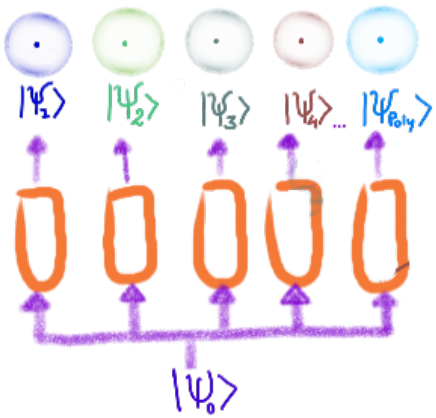


Figure: Each circuit corresponds to a distinct output physical state, each shown here with an ϵ-ball around

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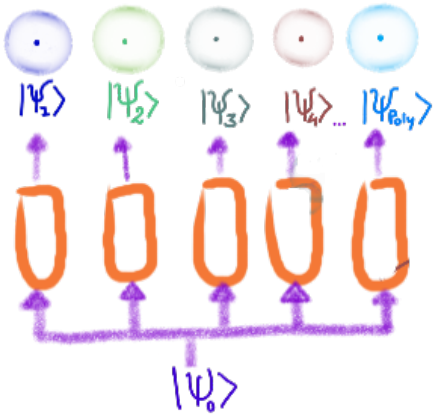


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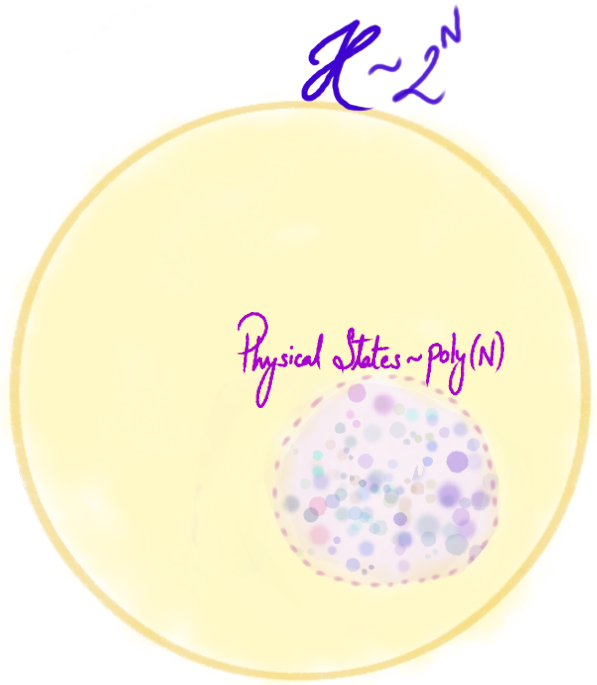


Figure: Union of ϵ-balls occupies a small region of $\mathcal{H}$

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Total Number of Quantum States

K-qubits live on a 2^K dimensional Hilbert Space.

$$\text{Vol}(K\text{-qubit } \mathcal{H}) = \text{Vol}\left(S^{2^{K+1}-1}\right), \quad \text{Vol}(\epsilon\text{-ball}) = \text{Vol}\left(S^{2^{K+1}-2}\right).$$

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$$\boxed{\frac{\text{Vol}(\text{Physical States})}{\text{Vol}(\text{Total States})} = \frac{N_c \cdot \text{Vol}(\epsilon\text{-ball})}{\text{Vol}(K\text{-qubit } \mathcal{H})} = O\left(K^K \, \epsilon^{2^K}\right)}$$

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Quantum Randomness: No-go theorem

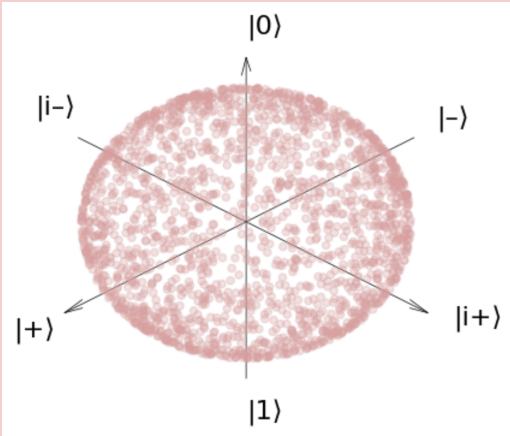


Figure: States drawn from a uniform probability measure over $\mathcal{H}$

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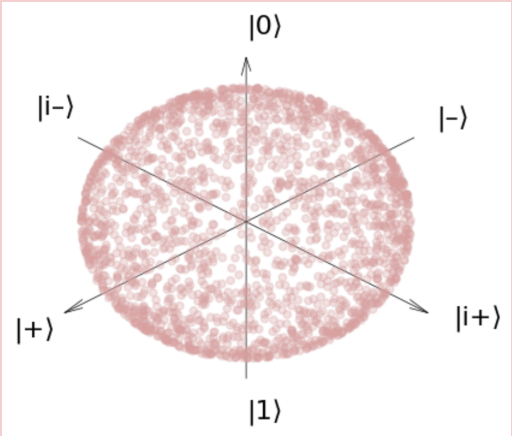


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It shows that Haar states are mathematical, not physically accessible, which in turn motivates the study of pseudorandomness generation as a substitute for true Haar randomness.

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  - Decoupling Time Ordering
  - Randomised sampling as opposed to previous deterministic methods.
- Folklore Theorem: Most states on the Hilbert space are physically inaccessible and thus do not represent naturally occuring states.
  - Proved for dynamics generated by arbitrary k-local Hamiltonians.

Example

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Probing non-equilibrium topological order on a quantum processor

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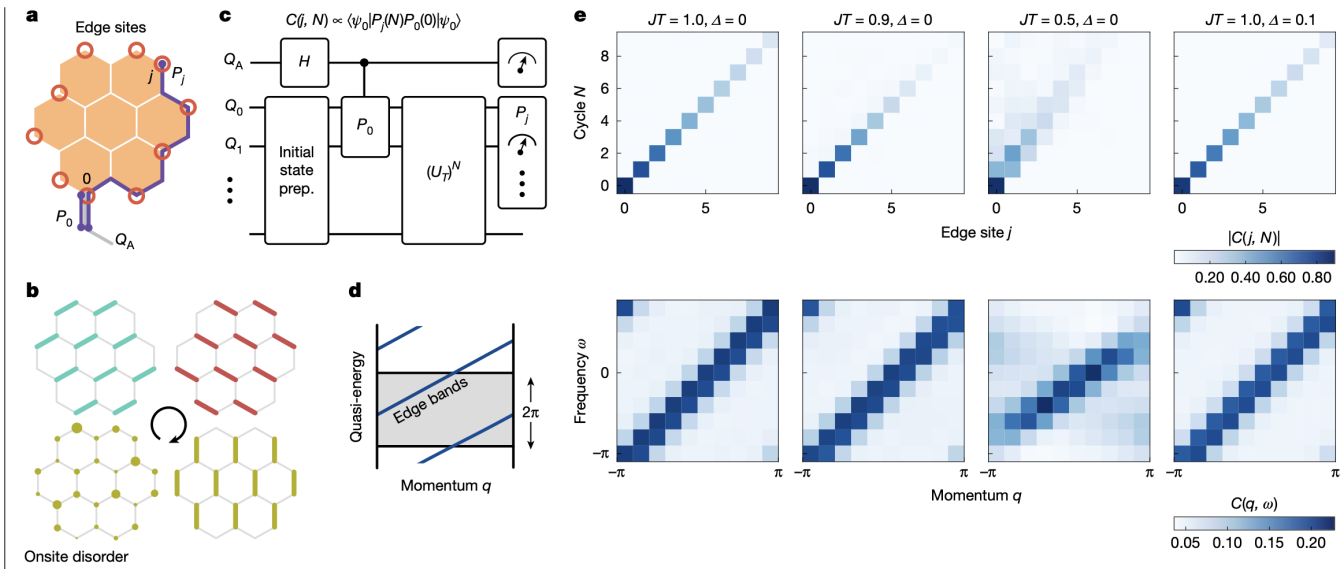


Fig. 3 | Majorana edge mode spectrum. a, The hexagonal lattice with edge sites shown in red. We also indicate the Pauli strings corresponding to the fermion occupation for two of the sites on the edge. **b**, Addition of a fourth time step with a random onsite Z-field that breaks integrability is given by $U_{\text{disorder}} = e^{-ijT\pi/4\sum_i h_i Z_i}$ with $h_i \in [-\Delta, \Delta]$. **c**, Schematic of the circuit to measure the unequal-time Pauli-string correlator, which is equivalent to the real-space Majorana spectral function, see text for more details. **d**, Exemplary sketch of spectrum in FTO phase. Edge states are shown in blue. **e**, The first row shows the spectral function in real space for $\Delta = 0$ and varying values of JT (first three columns), and for $JT = 1, \Delta = 0.1$ (last column), in which the data are averaged over $N_{\text{av}} = 20$ disorder configurations. The second row shows the Fourier-transformed data. Data are taken with dynamical decoupling, randomized compiling for CZ gates and post-selection on the middle plaquette in experiments without a disordered field (first three columns), see Methods ($N_{\text{twirling}} = 20$, $N_{\text{shots}} = 5 \times 10^5$, 27 qubits).